A Proof of a Conjecture of Ohba

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Abstract

We prove a conjecture of Ohba which says that every graph $G$ on at most $2\chi(G) + 1$ vertices satisfies $\chi_\ell(G) = \chi(G)$.

1 Introduction

List colouring is a variation on classical graph colouring. An instance of list colouring is obtained by assigning to each vertex $v$ of a graph $G$ a list $L(v)$ of available colours. An acceptable colouring for $L$ is a proper colouring $f$ of $G$ such that $f(v) \in L(v)$ for all $v \in V(G)$. When an acceptable colouring for $L$ exists, we say that $G$ is $L$-colourable. The list chromatic number $\chi_\ell$ is defined in analogy to the chromatic number:

$$\chi_\ell(G) = \min \{ k : G \text{ is } L\text{-colourable whenever } |L(v)| \geq k \text{ for all } v \in V(G) \}.$$

List colouring was introduced independently by Vizing [28] and Erdős et al. [4] and researchers have devoted a considerable amount of energy towards its study ever since (see e.g. [1, 27, 17, 29]).

A graph $G$ has an ordinary $k$-colouring precisely if it has an acceptable colouring for $L$ where $L(v) = \{1, 2, \ldots, k\}$ for all $v \in V(G)$. Therefore, the following bound is immediate:

$$\chi \leq \chi_\ell.$$

At first glance, one might expect the reverse inequality to hold as well. It would seem that having smaller intersection between colour lists could only make it easier to find an acceptable colouring. However, this intuition is misleading; in reality, a lack of shared colours can have the opposite effect.

To illustrate this phenomenon, consider the complete bipartite graph $K_{3,3}$ together with the list assignment exhibited in Figure 1. One can easily verify that no acceptable colouring exists, despite the fact that $K_{3,3}$ is bipartite and each list has size two.
Figure 1: A list assignment of $K_{3,3}$ with no acceptable colouring.

More generally, for $d \geq 2$ suppose that we assign to each vertex of $K_{\binom{2d-1}{d}, \binom{2d-1}{d}}$ a $d$-subset of a fixed $(2d-1)$-set, say $C$, such that every $d$-subset is assigned to exactly one vertex on each side of the bipartition. If an acceptable colouring exists for this list assignment, then the set of colours used on any given side of the bipartition must intersect every list. Since no set of size less than $d$ can have this property, any acceptable colouring must use at least $d$ colours on each side of the bipartition. However, no colour can be used on both sides and so any acceptable colouring must use at least $2d$ distinct colours from $C$, contradicting the fact that $|C| = 2d - 1$. This proves that

$$\chi_{\ell} \left( K_{\binom{2d-1}{d}, \binom{2d-1}{d}} \right) > d.$$  

Hence, there are bipartite graphs with arbitrarily large list chromatic number, and therefore $\chi_{\ell}$ is not bounded above by any function of $\chi$ (see also [1]). The problem of determining which graphs satisfy $\chi_{\ell} = \chi$ is well studied; such graphs are said to be chromatic-choosable [20].

The famous List Colouring Conjecture claims that every line graph is chromatic-choosable. It first appeared in print in a paper of Bollobás and Harris [2], but had also been formulated independently by Albertson and Collins, Gupta, and Vizing (see [8]). Galvin [5] showed that the List Colouring Conjecture is true for line graphs of bipartite graphs. In [13], Kahn proved that the List Colouring Conjecture is asymptotically correct, and his asymptotics were improved by Molloy and Reed in [18, 19].

\footnote{This insightful example first appeared in a paper of Erdős et al. [4].}
A more general conjecture of Gravier and Maffray \cite{GravierMaffray} says that every claw-free graph is chromatic-choosable. Since line graphs are claw-free, their conjecture would imply the List Colouring Conjecture. Gravier and Maffray proved their conjecture for the special case of elementary graphs.

Ohba \cite{Ohba} proved that $\chi_\ell(G + K_n) = \chi(G + K_n)$ for any graph $G$ and sufficiently large $n$, where $G + H$ denotes the join of $G$ and $H$. In their original paper \cite{ErdosEtAl}, Erdős et al. proved that the complete multipartite graph $K_{2,2,\ldots,2}$ is chromatic-choosable and the same was proved for $K_{3,2,2,\ldots,2}$ by Gravier and Maffray in \cite{GravierMaffray}. This paper concerns a conjecture of Ohba \cite{Ohba}, which implies the last three results.

**Conjecture 1.1 (Ohba \cite{Ohba}).** If $|V(G)| \leq 2\chi(G) + 1$, then $G$ is chromatic-choosable.

Infinite families of graphs satisfying $|V(G)| = 2\chi(G) + 2$ and $\chi_\ell(G) > \chi(G)$ are exhibited in \cite{Ohba} and so Ohba’s Conjecture is best possible.

It is easy to see that the operation of adding an edge between vertices in different colour classes of a $\chi(G)$-colouring does not increase $\chi$ or decrease $\chi_\ell$. It follows that Ohba’s Conjecture is true for all graphs if and only if it is true for complete multipartite graphs. Thus, we can restate Ohba’s Conjecture as follows.

**Conjecture 1.1 (Ohba \cite{Ohba}).** If $G$ is a complete $k$-partite graph on at most $2k + 1$ vertices, then $\chi_\ell(G) = k$.

This conjecture has attracted a great deal of interest and substantial evidence has been amassed for it. This evidence mainly comes in two flavours: replacing $2k + 1$ with a smaller function of $k$, or restricting to graphs whose stability number is bounded above by a fixed constant.

**Theorem 1.2.** Let $G$ be a complete $k$-partite graph. If any of the following are true, then $G$ is chromatic-choosable.

(a) $|V(G)| \leq k + \sqrt{2k}$ (Ohba \cite{Ohba});

(b) $|V(G)| \leq \frac{5}{3}k - \frac{4}{3}$ (Reed and Sudakov \cite{ReedSudakov});

(c) $|V(G)| \leq (2 - \varepsilon)k - n_0(\varepsilon)$ for some function $n_0$ of $\varepsilon \in (0, 1)$. (Reed and Sudakov \cite{ReedSudakov}).

\footnote{Recall that a graph is claw-free if it does not contain an induced copy of $K_{1,3}$.}

\footnote{A graph $G$ is elementary if its edges can be 2-coloured so that any induced path on three vertices contains edges of both colours.}

\footnote{Reed and Sudakov \cite{ReedSudakov} actually prove that there is a function $n_1(\varepsilon)$ such that if $n_1(\varepsilon) \leq |V(G)| \leq (2 - \varepsilon)k$, then $G$ is chromatic-choosable. The original statement is equivalent to our formulation here.}
Definition 1.3. A maximal stable set of a complete multipartite graph is called a part.

Theorem 1.4. Let $G$ be a complete $k$-partite graph on at most $2k+1$ vertices and let $\alpha$ be the size of the largest part of $G$. If $\alpha \leq 5$, then $G$ is chromatic choosable (Kostochka et al. [15]; He et al. [25] proved the result for $\alpha \leq 3$).

In this paper, we prove Ohba’s Conjecture. We divide the argument into three main parts. In Section 2 we show how a special type of proper non-acceptable colouring of $G$ can be modified to yield an acceptable colouring for $L$. Then in Section 3 we argue that, under certain conditions, it is possible to find a colouring of this type. Finally, in Section 4 we complete the proof by showing that if Ohba’s Conjecture is false, then there exists a counterexample which satisfies the conditions described in Section 3.

For the proper non-acceptable colourings we consider in Section 2, if $v$ is coloured with a colour $c$ not on $L(v)$, then we insist that $c$ appears only on $v$, and that $c$ appears on the lists of many vertices. This helps us to prove, in Section 2, that we can indeed modify such colourings to obtain acceptable colourings, as there are at least $k$ colours which can be used on $v$ and many vertices on which $c$ can be used.

In Sections 3 and 4, we combine Hall’s theorem and various counting arguments to prove that such colourings exist for minimal counterexamples to Ohba’s Conjecture. In the rest of this section we provide some properties of such minimal counterexamples, which will help us do so. In particular we show that for any minimal counterexample $G$, if $G$ is not $L$-colourable, then the total number of colours in the union of the lists of $L$ is at most $2k$. This upper bound on the number of colours, foreshadowed in earlier work of Kierstead and Reed and Sudakov, is crucial in that it implies the existence of colours which appear in the lists of many vertices, which our approach requires.

A variant of Ohba’s Conjecture for on-line list colouring has recently been proposed [11, 16]. We remark that this problem remains open, since our approach (in fact, any approach which depends heavily on Hall’s Theorem) does not apply to the on-line variant.

1.1 Properties of a Minimal Counterexample

Throughout the rest of the paper we assume, to obtain a contradiction, that Ohba’s Conjecture is false and we let $G$ be a minimal counterexample in the sense that $G$ is a complete $k$-partite graph on at most $2k+1$ vertices such that $\chi_{\ell}(G) > k$ and Ohba’s Conjecture is true for all graphs on fewer
than \(|V(G)|\) vertices. Throughout the rest of the paper, \(L\) will be a list assignment of \(G\) such that \(|L(v)| \geq k\) for all \(v \in V(G)\) and \(G\) is not \(L\)-colourable. Define \(C := \cup_{v \in V(G)} L(v)\) to be the set of all colours.

Let us illustrate some properties of a minimal counterexample. To begin, suppose that for a non-empty set \(A \subseteq V(G)\) there is a proper colouring \(g : A \rightarrow C\) such that \(g(v) \in L(v)\) for all \(v \in A\).

Such a mapping is called an acceptable partial colouring for \(L\). Define \(G' := G - A\) and \(L'(v) := L(v) - g(A)\) for each \(v \in V(G')\). If for some \(\ell \geq 0\) the following inequalities hold, then we can obtain an acceptable colouring of \(G'\) for \(L'\) by minimality of \(G\):

\[
|V(G')| \leq 2(k - \ell) + 1, \quad (1.1)
\]

\[
\chi(G') \leq k - \ell, \quad (1.2)
\]

\[
|L'(v)| \geq k - \ell \text{ for all } v \in V(G'). \quad (1.3)
\]

However, such a colouring would extend to an acceptable colouring for \(L\) by colouring \(A\) with \(g\), contradicting the assumption that \(G\) is not \(L\)-colourable. Thus, no such \(A\) and \(g\) can exist.

This argument can be applied to show that every part \(P\) of \(G\) containing at least two elements must have \(\cap_{v \in P} L(v) = \emptyset\). Otherwise, if \(c \in \cap_{v \in P} L(v)\), then the set \(A := P\) and function \(g(v) := c\) for all \(v \in A\) would satisfy (1.1), (1.2) and (1.3) for \(\ell = 1\), a contradiction. We have proven:

**Lemma 1.5.** If \(P\) is a part of \(G\) such that \(|P| \geq 2\), then \(\cap_{v \in P} L(v) = \emptyset\).

Many of our results are best understood by viewing an instance of list colouring in terms of a special bipartite graph. Let \(B\) be the bipartite graph with bipartition \((V(G), C)\) where each \(v \in V(G)\) is joined to the colours of \(L(v)\). For \(x \in V(G) \cup C\), we let \(N_B(x)\) denote the neighbourhood of \(x\) in \(B\). Clearly a matching in \(B\) corresponds to an acceptable partial colouring for \(L\) where each colour is assigned to at most one vertex. Recall the classical theorem of Hall [9] which characterizes the sets in bipartite graphs which can be saturated by a matching.

**Theorem 1.6** (Hall’s Theorem [9]). Let \(B\) be a bipartite graph with bipartition \((X, Y)\) and let \(S \subseteq X\). Then there is a matching \(M\) in \(B\) which saturates \(S\) if and only if \(|N_B(S')| \geq |S'|\) for every \(S' \subseteq S\).

In the next proposition, we use the minimality of \(G\) and Hall’s Theorem to show that \(B\) has a matching of size \(|C|\). Slightly different forms of this result appear in the works of Kierstead [14] and Reed and Sudakov [23].
Proposition 1.7. There is a matching in $B$ which saturates $C$.

Proof. Suppose to the contrary that no such matching exists. Then there is a set $T \subseteq C$ such that $|N_B(T)| < |T|$ by Hall’s Theorem. Suppose further that $T$ is minimal with respect to this property. Now, for some $c \in T$ let us define $S := T - c$ and $A := N_B(S)$. By our choice of $T$, we have that $|N_B(S')| \geq |S'|$ for every subset $S'$ of $S$. Thus, by Hall’s Theorem there is a matching $M$ in $B$ which saturates $S$. Moreover, we have

$$|A| \geq |S| = |T| - 1 \geq |N_B(T)| \geq |A|.$$ 

This proves that $|A| = |S|$ and, since $N_B(T)$ is non-empty, we have that $A$ is non-empty. In particular, $M$ must also saturate $A$. Let $g : A \to S$ be the bijection which maps each vertex of $A$ to the colour that it is matched to under $M$. Then clearly $g$ is an acceptable partial colouring for $L$. Since $A = N_B(S)$, every $v \in V(G) - A$ must have $L(v) \cap g(A) = L(v) \cap S = \emptyset$. Thus, $A$ and $g$ satisfy (1.1), (1.2) and (1.3) for $\ell = 0$, a contradiction.

Let us rephrase the above proposition into a form which we will apply in the rest of this section.

Corollary 1.8. There is an injective function $h : C \to V(G)$ such that $c \in L(h(c))$ for all $c \in C$.

The following is a simple, yet useful, consequence of this result.

Corollary 1.9. $|C| < |V(G)| \leq 2k + 1$.

Proof. If $|V(G)| \leq |C|$, then the injective function $h : C \to V(G)$ from Corollary 1.8 would be a bijection. The inverse of $h$ would be an acceptable colouring for $L$ since each colour $c \in C$ would appear on exactly one vertex of $G$ for which $c$ is available. This contradicts the assumption that $G$ is not $L$-colourable.

Corollary 1.10. If $u, v \in V(G)$ such that $L(u) \cap L(v) = \emptyset$, then $L(u) \cup L(v) = C$ and $|C| = 2k$.

Proof. Since the list of every vertex has size $k$ if two lists $L(u)$ and $L(v)$ are disjoint, then $|C| \geq |L(u) \cup L(v)| \geq 2k$. However, by Corollary 1.9 we have $|C| < |V(G)| \leq 2k + 1$ and so it must be the case that $L(u) \cup L(v) = C$ and $|C| = 2k$.

By Corollary 1.10 the difference between $|V(G)|$ and $|C|$ is always positive. Throughout the rest of the paper, it will be important for us to keep track of this quantity.
Definition 1.11. \( \gamma := |V(G)| - |C| \).

We conclude this section with two other useful consequences of Proposition 1.7.

Corollary 1.12. \( |V(G)| = 2k + 1 \).

Proof. If \( G \) has a part of size 2 then, by Lemma 1.5, the lists of the two vertices it contains are disjoint. Hence by Corollary 1.10, \(|C| = 2k\) and \(|V(G)| = 2k + 1\).

Otherwise \( G \) does not contain any part of size two and so \( G \) must contain a singleton part, say \( \{v\} \). If \(|V(G)| \leq 2k\), then we can obtain an acceptable colouring by using an arbitrary colour of \( L(v) \) to colour \( v \) and applying minimality of \( G \). The result follows.

Proposition 1.13. Let \( f : V(G) \to C \) be a proper colouring. Then there exists a surjective proper colouring \( g : V(G) \to C \) such that if \( g(v) \notin L(v) \), then \( g(v) = f(v) \).

Proof. Let \( h : C \to V(G) \) be an injective function such that \( c \in L(h(c)) \) for all \( c \in C \), which is guaranteed to exist by Corollary 1.8. Say that a proper colouring \( g : V(G) \to C \) agrees with \( h \) on \( v \in V(G) \) if \( h(g(v)) = v \). Choose \( g : V(G) \to C \) to be a proper colouring with the property that if \( g(v) \notin L(v) \), then \( g(v) = f(v) \) and subject to this maximizing the number of vertices on which \( g \) agrees with \( h \).

We will be done if we can prove that \( g \) is surjective. Suppose not, let \( c \in C - g(V(G)) \), and let \( w = h(c) \). Then clearly \( g \) does not agree with \( h \) on \( w \). Consider the function \( g' \) defined by

\[
g'(v) := \begin{cases} 
c & \text{if } v = w, 
g(v) & \text{otherwise}. \end{cases}
\]

Since \( c \in L(w) \), we have that \( g' \) retains the property that if \( g'(v) \notin L(v) \), then \( g'(v) = f(v) \). Moreover, \( g' \) is a proper colouring which agrees with \( h \) on \( w \) and on every vertex on which \( g \) agrees with \( h \). This contradicts our choice of \( g \). The result follows.

\(^5\)Otherwise, all parts of \( G \) would contain at least 3 elements and so \( 3k \leq |V(G)| \leq 2k + 1 \), which would imply \( k \leq 1 \). Ohba’s Conjecture is trivially true in this case.
2 Bad Colourings With Good Properties

In this section, we show that certain types of non-acceptable colourings can be modified to produce acceptable colourings. The following definitions describe the types of colourings that we are interested in. Say that a vertex \( v \in V(G) \) is a **singleton** if \( \{v\} \) is a part of \( G \).

**Definition 2.1.** A colour \( c \in C \) is said to be

- **globally frequent** if it appears in the lists of at least \( k+1 \) vertices of \( G \).
- **frequent among singletons** if it appears in the lists of at least \( \gamma \) singletons of \( G \).

If \( c \) is either globally frequent or frequent among singletons, then we say that \( c \) is **frequent**.

**Definition 2.2.** A proper colouring \( f : V(G) \to C \) is said to be **near-acceptable** for \( L \) if for every vertex \( v \in V(G) \) either

- \( f(v) \in L(v) \), or
- \( f(v) \) is frequent and \( f^{-1}(f(v)) = \{v\} \).

Suppose that \( f \) is a proper colouring of \( G \) and let \( V_f := \{f^{-1}(c) : c \in C\} \) be the set of colour classes under \( f \). Generalizing the construction of \( B \) in Section 1.1, we define \( B_f \) to be a bipartite graph on \((V_f, C)\) where each colour class \( f^{-1}(c) \in V_f \) is joined to the colours of \( \cap_{v \in f^{-1}(c)} L(v) \). A matching in \( B_f \) corresponds to a partial acceptable colouring for \( L \) whose colour classes are contained in \( V_f \). We use this observation to prove the following.

**Lemma 2.3.** If there is a near-acceptable colouring for \( L \), then there is an acceptable colouring for \( L \).

**Proof.** Suppose that there is a near-acceptable colouring \( f \) for \( L \). By Proposition 1.13 we may assume that \( f \) maps surjectively to \( C \). A matching in \( B_f \) which saturates \( V_f \) would indicate an acceptable colouring for \( L \) with the same colour classes as \( f \). Therefore, we assume that no such matching exists. By Hall’s Theorem, there is a set \( S \subseteq V_f \) so that \( |N_{B_f}(S)| < |S| \). We choose such an \( S \) which maximizes \( |S| - |N_{B_f}(S)| \) over all subsets of \( V_f \).

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6One can think of this construction as taking the graph \( G \) and collapsing each colour class of \( f \) into a single vertex. Each collapsed vertex is then assigned a list which is the intersection of the lists of all vertices in its corresponding colour class.
Since \(|N_{B_f}(S)| < |S|\) there must be a colour class \(f^{-1}(c^*) \in S\) so that \(c^* \notin N_{B_f}(S)\). In particular, we have \(c^* \notin N_{B_f}(f^{-1}(c^*))\). It follows that there is a vertex \(v\) so that \(f(v) = c^*\) and \(c^* \notin L(v)\). Since \(f\) is near-acceptable, we must have that \(c^*\) is frequent and \(f^{-1}(c^*) = \{v\}\).

**Case 1:** \(c^*\) is globally frequent.

Since \(f^{-1}(c^*) = \{v\} \in S\), we have that \(N_{B_f}(S) \supseteq N_{B_f}(f^{-1}(c^*)) = L(v)\) and so \(|N_{B_f}(S)| \geq k\). This implies that

\[
|S| > |N_{B_f}(S)| \geq k. \tag{2.1}
\]

However, since \(c^* \notin N_{B_f}(S)\) we have that every colour class of \(S\) must contain a vertex whose list does not contain \(c^*\). Since \(c^*\) is frequent, there are at most

\[
|V(G)| - (k + 1) \leq k
\]

such vertices. Thus, \(|S| \leq k\), contradicting \(2.1\) and so the result holds in this case.

**Case 2:** \(c^*\) is frequent among singletons.

We let \(\ell\) be the number of colour classes under \(f\) with more than one element. We let \(A\) be the union of the colour classes of \(f\) which either are not in \(S\) or contain more than one element. We will find a partial acceptable colouring \(g\) of \(A\) satisfying \((1.1)\), \((1.2)\) and \((1.3)\) for \(\ell\), thereby obtaining a contradiction.

Since \(A\) contains all colour classes of \(f\) with more than one element, \(A\) contains at least \(2\ell\) vertices and so \((1.1)\) holds for \(\ell\). Furthermore, for every singleton \(v\) with \(c^* \in L(v)\), \(\{v\}\) is a colour class of \(f\) which cannot be in \(S\), because \(c^* \notin N_B(S)\). Thus, every such singleton is in \(A\), and, since \(c^*\) is singleton frequent, we have that \(\chi(G - A) \leq k - \gamma\). Since \(f\) maps surjectively to \(C\), we have that \(\ell \leq \gamma\) and so \((1.2)\) holds for \(\ell\). We will insist that our partial acceptable colouring satisfies

\[
g(A) \text{ contains at most } \ell \text{ colours of } N_B(S). \quad (*)
\]

Now, for every vertex \(w \in V(G) - A\) we have that \(\{w\}\) is a colour class of \(f\) contained in \(S\), and hence \(L(w) \subseteq N_B(S)\). Thus, if \((\dagger)\) holds, then \((1.3)\) holds for \(\ell\).

Thus, to obtain a contradiction we need only specify how to obtain a partial acceptable colouring of \(A\) satisfying \((\dagger)\). To begin, we note that, since \(S\) was chosen to maximize \(|S| - |N_B(S)|\), for any \(T \subseteq V_f - S\) we
have $|N_B(T) - N_B(S)| \geq |T|$. It follows, by Hall’s Theorem, that there is a matching $M$ in $B - N_B(S)$ saturating $V_f - S$. This matching points out a partial acceptable colouring $h$ for the set $A_1$ of vertices in the colour classes of $V_f - S$ using only colours in $C - N_B(S)$. We note that $f$ is a partial acceptable colouring when restricted to the vertices in the colours classes in $S$ with more than one element, since if $f(v) \notin L(v)$ then $f^{-1}(f(v)) = \{v\}$. We define $g$ so that it agrees with $h$ on $A_1$ and agrees with $f$ on the rest of $A$. It is a partial acceptable colouring because, by their definitions, $g(A_1)$ is disjoint from $N_B(S)$ while $g(A - A_1)$ is contained in $N_B(S)$. Furthermore, since $A - A_1$ consists of the union of at most $\ell$ colour classes of $f$, we see that (*) holds for $g$ and we are done.

3 Constructing Near-Acceptable Colourings

In the previous section, we saw that finding an acceptable colouring for $L$ is equivalent to finding a near-acceptable colouring for $L$. However, in practice it can be much easier to construct a near-acceptable colouring than an acceptable colouring. In constructing a near-acceptable colouring, we need not worry about whether a frequent colour $c$ is available for a vertex $v$, provided that $v$ is the only vertex to be coloured with $c$. In this section, we exploit this flexibility to prove the following result.

Lemma 3.1. If $C$ contains at least $k$ frequent colours, then there is a near-acceptable colouring for $L$.

Proof. We let $F$ be a set of $k$ frequent colours and apply a three phase colouring procedure. In the first phase, we choose a colour class for each colour in $C - F$ so as to maximize the total number of vertices coloured by the resultant partial colouring. Subject to this, we maximize the number of parts containing a coloured vertex. In the second phase, we colour some portion of the remaining vertices with colours of $F$ which are available to them. We claim that our choice in the first phase allows us to carry out the second phase in such a way that when it terminates, the number of uncoloured vertices is at most the number of unused colours in $F$. This ensures that we can obtain a near-acceptable colouring in a straightforward third phase where we assign each unused colour in $F$ to at most one uncoloured vertex. To complete the proof it remains to prove the claim.

Case 1: There is a part of size 2, neither of whose vertices were coloured in Phase 1.
Let $P = \{u, v\}$ be such a part. By Lemma 1.5 we have that $L(u) \cap L(v) = \emptyset$ and so by Corollary 1.10 we have that $|C| = 2k$ and $L(u) \cup L(v) = C$. In particular, every colour $c \in C - F$ must be used on at least one vertex in Phase 1, for if not we could use $c$ to colour one of $u$ or $v$. Since $|C| = 2k$, we have $|C - F| = k$ which implies that at least $k$ vertices are coloured in the first phase.

If at least $k + 1$ vertices were coloured in the first phase, then the number of remaining vertices is at most $|V(G)| - (k + 1) = k = |F|$ and so we can move directly to the third phase. Therefore, we assume that exactly $k$ vertices were coloured in the first phase and every colour of $C - F$ is used to colour exactly one vertex. Since no vertex of $P$ is coloured, there must be a part $Q$ containing two coloured vertices, say $x$ and $y$. However, since $L(u) \cup L(v) = C$ we can uncolour $x$ and use its colour on one of $u$ or $v$, which increases the number of parts with a coloured vertex and contradicts our choice in Phase 1. This contradiction completes the proof in this case.

**Case 2:** Every part of size two contains a vertex which was coloured in Phase 1.

For each part $P$ let $R_P$ denote the set of vertices of $P$ which are not coloured in Phase 1. Label the parts of $G$ as $P_1, P_2, \ldots, P_k$ so that $|R_{P_1}| \geq |R_{P_2}| \geq \cdots \geq |R_{P_k}|$. For each part $P_i$, in turn, we try to colour $R_{P_i}$ with a frequent colour which has not yet been used and is available for every vertex of that set.

For some $i \geq 0$, suppose that $R_{P_1}, R_{P_2}, \ldots, R_{P_i}$ have been successfully coloured with $i$ colours of $F$. That is, the set $U$ of unused colours of $F$ has size $k - i$. If there are fewer than $k - i + 1$ vertices which have not yet been coloured, then we terminate Phase 2 and move to Phase 3. Otherwise, we continue the process by trying to colour $R_{P_{i+1}}$ with a colour in $U$.

Suppose that no colour of $U$ is available for every vertex of $R_{P_{i+1}}$. Since there are at least $k - i + 1$ vertices which are not yet coloured and at most $k - i$ parts which are not completely coloured, we must have $|R_{P_{i+1}}| \geq 2i$ by our choice of ordering. Thus, again by our choice of ordering, there are at least $|R_{P_{i+1}}| i \geq 2i$ vertices which have been coloured so far in the second phase. Since $|V(G)| \leq 2k + 1$ and at least $k - i + 1$ vertices have not been coloured, we have that there are at most $k + i$ vertices that have been coloured in the first two phases combined. Therefore, at most $k + i - 2i = k - i$ vertices were coloured in the first phase.

Let us show that exactly $k - i$ vertices were coloured in the first phase. To do so, we use the fact that every colour in $U$ is absent from $L(v)$ for at least one $v$ in $R_{P_{i+1}}$. Since there are $k$ colours available for each vertex of
\( R_{P_{t+1}} \) and exactly \( k \) colours in \( F \), these absences imply that the colours of 
\( C - F \) must appear at least \( |U| = k - i \) times in the lists of vertices of \( R_{P_{t+1}} \).
Now if a colour \( c \in C - F \) is available for \( j > 0 \) vertices of \( R_{P_{t+1}} \), then:

(i) \( c \) was not used to colour any vertex of \( P_{t+1} \). Otherwise, we could 
increase the number of vertices which receive colour \( c \) by using \( c \) to 
colour those \( j \) vertices of \( R_{P_{t+1}} \) for which it is available. This would 
contradict our choice of colouring in Phase 1.

(ii) at least \( j \) vertices were coloured with \( c \) in Phase 1. Otherwise, we could 
uncolour the vertices that were coloured with \( c \) in Phase 1 and use \( c \) 
to colour \( j \) vertices of \( R_{P_{t+1}} \) instead, again contradicting our choice of 
colouring in Phase 1.

Thus, since the colours of \( C - F \) appear at least \( k - i \) times in the lists 
of vertices in \( R_{P_{t+1}} \), by (i) and (ii) we have that at least \( k - i \) vertices of 
\( V(G) - P_{t+1} \) were coloured in the first phase. Since we have already proven 
that at most \( k - i \) vertices were coloured in the first phase, this implies

\[
\text{exactly } k - i \text{ vertices are coloured in Phase 1, and} \quad (3.1) \\
\text{none of these vertices are in } P_{t+1}, \text{ i.e. } R_{P_{t+1}} = P_{t+1}. \quad (3.2)
\]

However, recall that we are assuming that every part of size two contains 
a vertex which is coloured in Phase 1. Since no vertex of \( P_{t+1} \) is coloured 
in Phase 1, we must have \( |R_{P_{t+1}}| = |P_{t+1}| \geq 3 \). This implies that at least 
\( |R_{P_{t+1}}| i \geq 3i \) vertices were coloured in the second phase, and so at most 
\( k + i - 3i = k - 2i \) vertices were coloured in the first phase. Thus, by (3.1) 
we must have \( k - 2i \geq k - i \) which implies \( i = 0 \).

Therefore, we have that exactly \( k \) vertices are coloured in the first phase 
and no vertices have been coloured so far in the second phase. That is, we 
have that \( U = F \), exactly \( k + 1 \) vertices remain uncoloured, and \( R_{P_1} = P_1 \) 
by (3.2). We will be done if we can find a colour \( c \in F \) which is available for 
two vertices \( u, v \in P_1 \). If such a colour exists, then we simply colour \( u \) and 
v with \( c \) which reduces the number of remaining vertices to \( k - 1 \), which is 
precisely the number of unused colours of \( F \).

Recall that the number of times the colours of \( C - F \) appear in lists of 
vertices in \( P_1 \) is at most the number of vertices coloured in the first phase, 
which is exactly \( k \). Since \( |P_1| \geq 3 \) and each list has size at least \( k \), we have 
that the colours of \( F \) must appear at least \( 2k \) times in the lists of vertices in 
\( P_1 \). Since \( |F| = k \), there is a colour in \( F \) (in fact, many) which is available 
for at least two vertices in \( P_1 \). This completes the proof.
4 Adding Colours to the Lists

From now on, we impose an additional requirement on our list assignment \( L \) for which there is no acceptable colouring. We insist that it is maximal in the sense that increasing the size of any list makes it possible to find an acceptable colouring. That is, for any \( v \in V(G) \) and \( c \in C - L(v) \), if we define \( L^*(v) = L(v) \cup \{c\} \) and \( L^*(u) = L(u) \) for all \( u \neq v \), then there is an acceptable colouring for \( L^* \). Given this property, it is straightforward to prove that every frequent colour is available for every singleton.

**Lemma 4.1.** If \( c \in C \) is frequent, then \( c \in L(v) \) for every singleton \( v \) of \( G \).

**Proof.** Otherwise, add \( c \) to the list of \( v \). Since \( L \) is maximal, there is an acceptable colouring for this modified list assignment. Since \( G \) is not \( L \)-colourable, this colouring must use \( c \) to colour \( v \) and, since \( v \) is a singleton, \( v \) is the only vertex coloured with \( c \). Therefore, this colouring is a near-acceptable colouring for \( L \) and so by Lemma 2.3 we have that \( G \) is \( L \)-colourable, a contradiction. \( \square \)

Recall by Lemmas 2.3 and 3.1 that there are fewer than \( k \) frequent colours. We show now that this implies that there are at least \( \gamma \) singletons.

**Definition 4.2.** Let \( b \) denote the number of non-singleton parts of \( G \).

**Proposition 4.3.** \( G \) contains at least \( \gamma \) singletons.

**Proof.** Suppose to the contrary that the number of singletons, namely \( k - b \), is less than \( \gamma \). Let \( F' \) denote the set of all globally frequent colours. Then each colour of \( C - F' \) is available for at most \( k \) vertices, and each colour of \( F' \) is available for at most \( |V(G)| - b \leq 2k + 1 - b \) vertices by Lemma 1.5.

Thus,

\[
k|V(G)| \leq \sum_{v \in V(G)} |L(v)| = \sum_{c \in C} |N_B(c)| \\
\leq k|C - F'| + (2k + 1 - b)|F'| = k|C| + (k + 1 - b)|F'|.
\]

In other words,

\[
|F'| \geq \frac{k(|V(G)| - |C|)}{k + 1 - b} = \frac{k\gamma}{k + 1 - b}. \tag{4.1}
\]

Now, since we are assuming that \( k - b + 1 \leq \gamma \), we obtain \( |F'| \geq k \) by the above inequality. However, this contradicts the fact that there are fewer than \( k \) frequent colours. Thus, \( G \) must contain at least \( \gamma \) singletons. \( \square \)
Corollary 4.4. A colour \( c \in C \) is frequent if and only if it is available for every singleton.

Proof. By Proposition 4.3, any colour which is available for every singleton is automatically frequent among singletons. Therefore, by Lemma 4.1 we have that a colour is frequent if and only if it is available for every singleton. The result follows.

By the above lemma, to obtain the desired contradiction, we need only show that there are \( k \) colours which are available for every singleton. In fact, as the next result shows, it is enough to prove that there are at least \( b \) such colours.

Lemma 4.5. There are fewer than \( b \) frequent colours.

Proof. Suppose that there are at least \( b \) frequent colours and let \( A_b = \{c_1, c_2, \ldots, c_b\} \) be a set of \( b \) such colours. By Lemma 4.1 all such colours are available for every singleton. Label the singletons of \( G \) by \( v_{b+1}, v_{b+2}, \ldots, v_k \). For each \( i \in \{b+1, b+2, \ldots, k\} \), in turn, choose a colour \( c_i \in L(v_i) - A_{i-1} \) greedily and define \( A_i := A_{i-1} \cup \{c_i\} \). Let \( L' \) be a list assignment of \( G \) defined by

\[
L'(v) := \begin{cases} A_k & \text{if } v \text{ is a singleton,} \\ L(v) & \text{otherwise.} \end{cases}
\]

Clearly \( L' \) assigns each singleton to the same list of \( k \) colours. Hence, there are at least \( k \) frequent colours under \( L' \) and so by Lemmas 2.3 and 3.1 there is an acceptable colouring \( f' \) for \( L' \). We use this to construct an acceptable colouring \( f \) for \( L \), contradicting the fact that \( G \) is not \( L \)-colourable.

We let \( S \) denote the set of all singletons and for each \( v \in V(G) - S \) we set \( f(v) = f'(v) \). We note that \( f'(S) \) is a set of exactly \( k - b \) colours of \( A_k \), disjoint from \( f'(V(G) - S) = f(V(G) - S) \). We let

\[
S' := \{v_i \in S : c_i \in f'(S)\}
\]

We note that \( |S'| = k - b - |f'(S) \cap A_b| \) and hence \( |S - S'| = |f'(S) \cap A_b| \). For each \( v_i \in S' \), we set \( f(v_i) = c_i \). We arbitrarily choose a bijection \( \pi : S - S' \to f(S) \cap A_b \) and set \( f(v) = \pi(v) \) for every \( v \in S - S' \). Since each colour in \( A_b \) is available on every singleton we are done.

Before moving on to the final counting arguments, we establish two simple consequences of Proposition 4.3.

Corollary 4.6. \( G \) contains no part of size 2.
Proof. If $G$ contains a part $P = \{u, v\}$, then $L(u) \cap L(v) = \emptyset$ by Lemma 1.5. By Corollary 1.10, this implies that $|C| = 2k$ and so $\gamma = 1$. Thus, by Proposition 4.3 we have that $G$ contains a singleton $v$ and every colour of $L(v)$ is frequent among singletons since $\gamma = 1$. This contradicts the fact that there are fewer than $k$ frequent colours, and proves that $G$ contains no part of size two.

Corollary 4.7. $b \leq \frac{k+1}{2}$.

Proof. Since $G$ contains no parts of size 2 we have that $G$ consists of $k - b$ singletons and $b$ parts of size at least three. Therefore

$$(k - b) + 3b \leq |V(G)| \leq 2k + 1$$

which implies that $b \leq \frac{k+1}{2}$. □

4.1 A Bit of Counting

The final step is to apply a counting argument to show that there are at least $b$ colours which are frequent among singletons, which would contradict Lemma 4.5 and complete the proof of Ohba’s Conjecture. In order to do so, we will find a fairly large set $X$ of singletons such that $N_B(X)$ is fairly small. This implies that the average number of singletons on which a colour in $N_B(X)$ appears is large. Using this, we will show that there are at least $b$ colours in $N_B(X)$ which are available on $\gamma$ singletons, which gives us the desired contradiction. We begin with the following proposition.

Proposition 4.8. Let $c^*$ be a colour which is not available for every singleton. Then there is a set $X$ (depending on $c^*$) of singletons such that

(a) $|X| \geq k - b - \gamma + 1$, and

(b) $|\cup_{v \in X} L(v)| \leq 2k - |N_B(c^*)|$.

Proof. To prove this result, we modify a colouring which is almost acceptable as in the proof of Lemma 2.3. To begin, we obtain a proper colouring $f$ such that $f(v) \in L(v)$ for all $v$ except for one singleton coloured with $c^*$. To this end, we let $x$ be a singleton such that $c^* \notin L(x)$ and define a list assignment $L^*$ by

$L^*(v) = \begin{cases} L(x) \cup \{c^*\} & \text{if } v = x, \\ L(v) & \text{otherwise.} \end{cases}$

Then, since $L$ is maximal there is an acceptable colouring $f$ for $L^*$. Clearly $f^{-1}(c^*) = \{x\}$ and for every $v \in V(G) - x$ we have $f(v) \in L(v)$. Clearly
By Proposition 1.13, we can also assume that \( f \) is surjective. If there is a matching in \( B_f \) which saturates \( V_f \), then \( G \) is \( L \)-colourable. Thus, we obtain a set \( S \subseteq V_f \) such that \( |N_{B_f}(S)| < |S| \).

In particular, since \( c \in N_{B_f}(f^{-1}(c)) \) for every \( c \neq c^* \) it must be the case that \( f^{-1}(c^*) \in S \) and \( c^* \notin N_{B_f}(S) \). Since \( c^* \notin N_{B_f}(S) \), every colour class of \( S \) must contain a vertex whose list does not contain \( c^* \). Thus, we have

\[
|N_{B_f}(S)| < |S| \leq |V(G)| - |N_{B_f}(c^*)| \leq 2k + 1 - |N_{B_f}(c^*)|.
\]

Now, define \( X \) to be the set of all singletons of \( G \) whose colour classes under \( f \) belongs to \( S \). Then we have \( \bigcup_{v \in X} L(v) \subseteq N_{B_f}(S) \), and so (b) holds.

Finally, if \( X \) contains fewer than \( k - b - \gamma + 1 \) singletons, then there are at least \( \gamma \) singletons whose colour class is contained in \( V_f - S \). This would allow us to proceed by the argument in Case 2 of Lemma 2.3 to obtain an acceptable colouring for \( L \). Thus, \( |X| \geq k - b - \gamma + 1 \). \( \square \)

We let \( c^* \) be a colour which is not frequent (equivalently, not available on every singleton), and subject to this maximizing \( |N_{B_f}(c^*)| \). Let \( X \) be a set of singletons as in Proposition 4.8.

Let \( Z \) be a set of \( b - 1 \) colours that appear most often in the lists of vertices of \( X \) and define \( Y := N_{B_f}(X) - Z \). That is, \( |Z| = b - 1 \) and if \( c_1 \in Z \) and \( c_2 \in Y \), then \( |N_{B_f}(c_1) \cap X| \geq |N_{B_f}(c_2) \cap X| \). We can further assume that every frequent colour appears in \( Z \), since by Lemma 4.1 these colours appear in the list of every vertex of \( X \) and by Lemma 4.5 there are at most \( b - 1 \) of them. Finally, let \( c' \in Y \) such that \( |N_{B_f}(c') \cap X| \) is maximized over all such colours. Our goal is to prove that \( |N_{B_f}(c') \cap X| \geq \gamma \), contradicting the assumption that \( Z \) contains every frequent colour. The next proposition describes one way that this can be done.

**Proposition 4.9.** If \( |N_{B_f}(c^*)| \geq k - \frac{(b-2\gamma+1)(b-1)}{\gamma} \), then \( |N_{B_f}(c') \cap X| \geq \gamma \).

**Proof.** Since \( |Z| = b - 1 \), every vertex \( v \in X \) must satisfy \( |L(v) \cap Y| \geq k - b + 1 \). Thus, we have

\[
|Y||N_{B_f}(c') \cap X| \geq \sum_{c \in Y} |N_{B_f}(c) \cap X| = \sum_{v \in X} |L(v) \cap Y| \geq |X|(k - b + 1).
\]

Combining this inequality with the two bounds of Proposition 4.8, we obtain the following:

\[
|N_{B_f}(c') \cap X| \geq \frac{|X|(k - b + 1)}{|Y|} \geq \frac{|X|(k - b + 1)}{|N_{B_f}(X)| - b + 1}
\]

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\begin{align*}
\quad \geq \frac{(k - b - \gamma + 1)(k - b + 1)}{2k - |N_B(c^*)| - b + 1}.
\end{align*}

If \( |N_B(c^*)| \geq k - \frac{(k-b-\gamma+1)(k-b+1)}{\gamma} \), then we obtain

\begin{align*}
|N_B(c') \cap X| \geq \frac{\gamma(k - b - \gamma + 1)(k - b + 1)}{\gamma(k - b + 1) + (k - b - 2\gamma + 1)(k - b + 1)} = \gamma.
\end{align*}

The result follows.

Our final goal will be to prove that \( |N_B(c^*)| \geq k - \frac{(k-b-\gamma+1)(k-b+1)}{\gamma} \). The proof requires some simple counting which is broken into two parts. First we prove a preliminary bound, and then we show that this bound is indeed large enough to invoke Proposition 4.9.

**Proposition 4.10.** \( |N_B(c^*)| \geq k - \frac{(b-1)(k+1-b)-k\gamma}{2k-\gamma-b+2} \).

**Proof.** Let \( F \) denote the set of all frequent colours. Recall that \( c^* \) was chosen to maximize \( |N_B(c^*)| \) over all colours which are not frequent. Thus, each colour \( c \notin F \) must have \( |N_B(c)| \leq |N_B(c^*)| \). Moreover, by Lemma 4.5 there are at most \( b - 1 \) frequent colours and by Lemma 1.5 every colour \( c \in C \) satisfies \( |N_B(c)| \leq |V(G)| - b \). We have \( |V(G)| = 2k + 1 \) by Corollary 1.12 and so,

\begin{align*}
(2k + 1)k \leq \sum_{v \in V(G)} |L(v)| &\leq \sum_{c \in C} |N_B(c)| \leq |C - F||N_B(c^*)| + |F|(2k + 1 - b) \\
&\leq (2k + 1 - \gamma - b + 1)|N_B(c^*)| + (b - 1)(2k + 1 - b).
\end{align*}

Solving for \( |N_B(c^*)| \), we obtain

\begin{align*}
|N_B(c^*)| &\geq \frac{(2k + 1)k - (b - 1)(2k + 1 - b)}{2k - \gamma - b + 2} \\
&= k + \frac{(\gamma + b - 1)k - (b - 1)(2k + 1 - b)}{2k - \gamma - b + 2} \\
&= k + \frac{k\gamma - (b - 1)(k + 1 - b)}{2k - \gamma - b + 2}
\end{align*}

as desired. \( \square \)

We use the above proposition to prove the following which, when combined with Proposition 4.9 completes the proof of Ohba’s Conjecture.

**Proposition 4.11.** \( |N_B(c^*)| \geq k - \frac{(k-b-2\gamma+1)(k-b+1)}{\gamma} \).
Proof. By Proposition 4.10 it suffices to prove
\[
k - \frac{(b - 1)(k + 1 - b) - k\gamma}{2k - \gamma - b + 2} \geq k - \frac{(k - b - 2\gamma + 1)(k + 1 - b)}{\gamma}
\]
or equivalently
\[
\frac{(k - b - 2\gamma + 1)(k + 1 - b)}{\gamma} \geq \frac{(b - 1)(k + 1 - b) - k\gamma}{2k - \gamma - b + 2}.
\]
(4.2)

Setting \(\eta := \frac{\gamma}{k + 1 - b} > 0\), rewriting \(\gamma\) as \(\eta(k + 1 - b)\), and dividing each side of (4.2) by the positive number \(k + 1 - b\) we see that it suffices to show that the following inequality is satisfied:
\[
\frac{1 - 2\eta}{\eta} \geq \frac{b - 1 - \eta k}{(k + 1) + (1 - \eta)(k + 1 - b)}.
\]
(4.3)

By (4.1), the set \(F'\) of globally frequent colours satisfies \(|F'| \geq k\eta\), and by Lemma 4.5 we have \(|F'| \leq b - 1\). Thus, we obtain \(b - 1 - \eta k \geq 0\) and \(\eta \leq \frac{b - 1}{k}\). Hence, by Corollary 4.7 \(\eta < \frac{1}{2}\). By Corollary 4.7 we have that \(b - 1 < \frac{k}{2}\) and \(k + 1 - b \geq \frac{k + 1}{2}\). Putting these together, we obtain a bound on the righthand side of (4.3):
\[
\frac{b - 1 - \eta k}{(k + 1) + (1 - \eta)(k + 1 - b)} < \frac{(\frac{1}{2} - \eta) k}{(\frac{3}{2} - \eta) (k + 1)} < \frac{1 - 2\eta}{3 - \eta} < \frac{1 - 2\eta}{\eta}.
\]
This establishes the bound in (4.3) and completes the proof. \(\square\)

This completes the proof of Ohba’s Conjecture.

5 Conclusion

As noted in the introduction, there are examples which show that, in general, this result cannot be extended to the case where \(|V(G)| \leq 2\chi(G) + 2\) (for example, see Figure 1). However, there are still interesting problems in this direction for complete multipartite graphs. One rather ambitious problem could be to characterize all complete \(k\)-partite graphs with \(\chi_\ell(G) = k\). Short of this, it would be interesting to characterize all such graphs on at most \(f(k)\) vertices for some function \(f\) which is larger than \(2k + 1\). Another direction could be to determine a function \(g_\ell(k)\) such that any graph with \(\chi(G) = k\) and \(|V(G)| \leq g_\ell(k)\) satisfies \(\chi_\ell(G) \leq k + \ell\). Certainly our result implies that the function \(g_\ell(k) = 2(k + \ell) + 1\) would suffice, but this is unlikely to be tight for \(\ell \geq 1\).
References


[21] , *Choice number of complete multipartite graphs with part size at most three*, Ars Combin. 72 (2004), 133–139.


